Grade 6 Math Circles

February 5th - February 9th, 2024

Number Theory: Divisibility

Section 1: Number Theory

At some point in our lives we were all taught how to perform division between two numbers, but it is not so common to learn how to quickly identify if a number divides another number. Considering the expression below, how could we possibly look at this expression and immediately determine that 3 is in fact a factor of 1641?

$$3\overline{)\,1641}$$

Figure 1: Long Division

Number theory is the study of the properties and principles that give order to numbers, specifically whole numbers! Today we will look at the concept of divisibility and how to apply its properties to quickly identify the possible factors/divisors of a given number!!

Stop and Think

What are all the positive whole number divisors of 8, 36, 15, and 7?

Section 2: Divisibility

Definition 1

Let x and y be whole numbers. We say x divides y and write $x \mid y$ if we can find a whole number z so that $y = x \times z$.

Exercise 2.1

Look at the following statements and identify if they are True or False using the definition of divisibility:

1. $3 \mid 27$
2. $4 \mid 32$
3. $2 \mid 51$
4. $16 \mid 7$
5. $5 \mid 133$
6. $9 \mid 81$

Factoring

In the exercise we just went through we saw how we can write numbers in terms of their positive whole number divisors. As a separate example consider 24 which we know can be written as 2×12, 6×4, 3×8 and 24×1 in doing this we can see that 24 is divisible by all the numbers we just listed, we say that 1, 2, 3, 4, 6, 8, 12 and 24 are all *factors* of 24. In a similar way we know that 1, 2, 5, 10 are factors of 10. Since 1 and 2 are both factors of 10 and 24, we say that 1 and 2 are *common factors* of 10 and 24.

Exercise 2.2

Find all the common factors of the following pairs of numbers

1. 8 and 24
2. 7 and 12
3. 14 and 21

We'll quickly come to see that common factors are our best friends, since they will allow us to simplify operations by "factoring", let's see this in an example;

Example 2.1

Factor the expression $24 + 10$ by a factor greater than 1.

Solution: When asked to factor an expression what we are actually begin asked to do is to divide an expression by a common factor without actually solving the expression given to us. Let's break this process up into some steps;

1. The first thing we will do is identify the factors of both numbers. From earlier we know that 1, 2, 3, 4, 6, 8, 12 and 24 are all the positive factors of 24, and 1, 2, 5, 10 are all the positive factors of 10. From these we identify that 1 and 2 are the common factors of 10 and 24.
2. Now we will write both 10 and 24 in terms of their factor of 2, this gives us $10 = 2 \times 5$ and $24 = 2 \times 12$.
3. Using our expressions we can now write $24 + 10 = (2 \times 12) + (2 \times 5)$.
4. Now that we have $(2 \times 12) + (2 \times 5)$ what we do is"pull out" the two from both sets of brackets giving us $(2 \times 12) + (2 \times 5) = 2(12 + 5)$.

Exercise 2.3

Factor the following expressions by the given common factor.

1. $8 + 24$ with common factor 4
2. $25 - 10$ with common factor 5

Properties of Divisibility

Now that we have formalized the definition of divisibility and gotten some work with factoring, let's explore divisibility in a bit more detail.

Exercise 2.4

We know that $4 \mid 12$ and $4 \mid 8$. We also know that $4 \mid 20$, but notice $20 = 12 + 8$. Let's explore this relationship together!

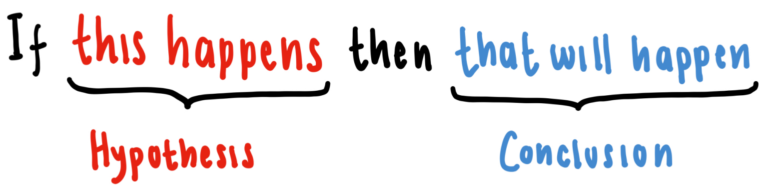
Find examples of three whole numbers a, b and c such that $c \mid a$ and $c \mid b$. Is it always the case that $c \mid (a + b)$?

As we saw from our exercise, this property seems to hold FOR ALL whole numbers!! This discovery will be called a conjecture, and what we want to do is prove that it is true.

Property 1 of Divisibility

If there exist whole numbers a, b and c such that $c \mid a$ and $c \mid b$, then $c \mid (a + b)$ and $c \mid (a - b)$.

The Language of Proofs (*if-then* statements): In mathematics, a mathematical proof is an argument which convinces others that a statement is true. Mathematical statements are commonly formatted as *if-then* statements which take the form below. For some very special mathematical



statements (like our divisibility rules) we'll see that the *converse* of a statement can be true as well, where the converse is the statement given by replacing the roles of the hypothesis and conclusion, to get a statement of the form "if *conclusion* then *hypothesis*".

In order to prove a statement like this we start by assuming that the *hypothesis* of the statement is true, and then use our knowledge of the mathematics presented in the hypothesis to build towards the *conclusion*! Let's try this out with an example.

Example 2.2

Prove that if there exist whole numbers a, b and c such that $c \mid a$ and $c \mid b$, then $c \mid (a + b)$.

Solution: Assume that there exists whole numbers a, b and c such that $c \mid a$ and $c \mid b$. Then by our definition of divisibility since $c \mid a$ it follows that there exists a whole number x such that $a = c \times x$. Similarly $b = c \times y$ for some whole number y. From our earlier work with factoring we can see that c is a common factor of a and b, and so we know that by factoring the expression $a + b$ by the common factor of c we get the following; $a + b = c \times (x + y)$, from this we immediately get that $c \mid (a + b)$ as desired.

**Note:* The same argument follows for the conclusion that $c \mid (a - b)$ by replacing all instances of $+$ with an instance of $-$.

Exercise 2.5

We know that $2 \mid 4$, we also know that $4 \mid 16$, notice that $2 \mid 16$ Let's explore this relationship together!

Find examples of three whole numbers a, b and c such that $a \mid b$ and $b \mid c$ Is it always the case that $a \mid c$?

Once again, we were not able to find an example in which this property does not hold, so lets make this conjecture and try to prove it!

Property 2 of Divisibility

If there exist whole numbers a, b and c such that $a \mid b$ and $b \mid c$, then $a \mid c$.

Example 2.3

Prove that if there exist whole numbers a, b and c such that $a \mid b$ and $b \mid c$, then $a \mid c$.

Solution: Assume that there exists whole numbers a, b and c such that $a \mid b$ and $b \mid c$.

- Since $a \mid b$ then the definition of divisibility tells us that there exists a whole number x such that $a \times x = b$.
- Similarly, since $b \mid c$ then the definition of divisibility tells us that there exists a whole number y such that $b \times y = c$.

Since $a \times x = b$ and $b \times y = c$ then $a \times (x \times y) = c$, and since x and y are both whole numbers, then their product will also be a whole number. Therefore, there exists a whole number $x \times y$ such that $a \times (x \times y) = c$ thus $a \mid c$ as desired.

Exercise 2.6

We know that $2 \mid 4$ if we multiply 4 by 3 we get 12 and we know that $2 \mid 12$, lets repeat this one last time, if we multiply 12 by 5 we get 60, and we know that $2 \mid 60$. Let's explore this relationship together!

Find examples of three whole numbers a, b and c such that $a \mid b$, Is it always the case that $a \mid (b \times c)$?

Once again this property comes from the fact that numbers share common factors, like in the example we did together, 2 is a factor of 4, 2 is a factor or 12, and 2 is a factor of 60! We thus have the following property...

Property 3 of Divisibility

If there exist whole numbers a and b such that $a \mid b$ then $a \mid (b \times c)$

Exercise 2.7

Prove that if there exist whole numbers a and b such that $a \mid b$ then $a \mid (b \times c)$

Section 3: Divisibility Rules

Now that we are equipped with the formal definition of divisibility, as well as Properties 1, 2 and 3, we have more than enough tools to begin learning, proving, and using some of the hidden secrets division has to offer! Not only are these rules easy to use but the converses of all these rules are true!

Rule for Division by 1

1 divides all numbers.

Rule for Division by 2

Let x be a whole number. If the ones digit of x is 0, 2, 4, 6, or 8 then $2 \mid x$

Decomposing Numbers Base 10

Recall that digits are the numbers 0 through 9 which build up all numbers, and we are able to build numbers by adding together the products of a number's digits times its place value for example $569 = (5 \times 100) + (6 \times 10) + (9 \times 1)$. Understanding Base 10 decomposition will come in handy for a lot of the proofs coming up, since many of the divisibility rules depend on the divisibility of a specific place value.

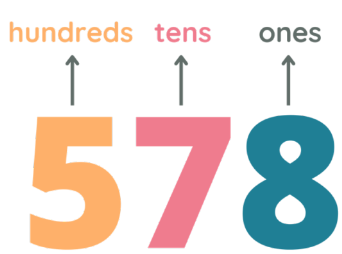


Figure 2: Digits

Exercise 3.1

Write 54823 as a sum of the products of each digit times its place value

Example 3.1

Prove that if the ones digit of a positive three digit whole number x is 0, 2, 4, 6, or 8, then $2 \mid x$.

Solution:

As usual we begin by assuming the hypothesis is true; so let's assume that the ones digit of a positive three digit whole number x is 0, 2, 4, 6, or 8. We are given 2 key pieces of information both of which we need to make use of.

Firstly, we are told that x is a three digit whole number so x is of the form "abc" where the whole number a is the hundreds digit, the whole number b is the tens digit, and the whole number c is the ones digit. Using base 10 decomposition we can write x as $x = (a \times 100) + (b \times 10) + (c \times 1)$.

To show that $2 \mid x$ we can show that $2 \mid (a \times 100)$, $2 \mid (b \times 10)$ and $2 \mid (c \times 1)$ so we can then use Property 1 of Divisibility to show that $2 \mid (a \times 100) + (b \times 10) + (c \times 1)$. Lets look at each of these statements individually;

1. To show that $2 \mid (a \times 100)$ we apply Property 3 of Divisibility, since $2 \times 50 = 100$ then $2 \mid 100$ so $2 \mid (a \times 100)$.
2. To show that $2 \mid (b \times 10)$ we apply Property 3 of Divisibility, since $2 \times 5 = 10$ then $2 \mid 10$ so $2 \mid (b \times 10)$.
3. To show that $2 \mid (c \times 1)$ we use the second piece of key information given to us in the hypothesis, that is that the ones digit of x is 0, 2, 4, 6, or 8, so c is one of 0, 2, 4, 6, or 8, and we know that 2 divides all of these numbers so $2 \mid (c \times 1)$.

Therefore $2 \mid (a \times 100)$, $2 \mid (b \times 10)$ and $2 \mid (c \times 1)$, so by Property 1 of Divisibility we have that $2 \mid (a \times 100) + (b \times 10) + (c \times 1)$, which gives us that $2 \mid x$ as desired.

Stop and Think

How would you prove that if a number is divisible by 2 then the ones digit of a number is 0, 2, 4, 6, or 8.

Stop and Think

Which of the following numbers are divisible by 3; 45, 34, 27, 15, 18, 81, 182? What do you notice about the digit sums of these numbers? What do you think could be the Rule for Division by 3?

Rule for Division by 3

Let x be a whole number. if the digits of x add up to a multiple of 3 then $3 \mid x$.

Exercise 3.2

Use the Rule for Division by 3 to determine if the following are true or false- check your answers with a calculator.

1. $3 \mid 27$
2. $3 \mid 300$
3. $3 \mid 200$
4. $3 \mid 28173$

Proving this rule seems tricky but it fully relies on the Base 10 Decomposition of a number. Lets look at an example first.

Example 3.2

Show that since $3 \mid (4 + 5)$ then $3 \mid 45$.

Solution:

We know that $4 + 5 = 9$ so $3 \mid (4 + 5)$. Our goal is to show that $3 \mid 45$. We'll do this by first writing 45 as a sum of the products of its digits times its place value, doing this we get $45 = (4 \times 10) + (5 \times 1)$. This is now where we'll use a very clever trick! We know that $4 \times 10 = (4 \times 9) + 4$, so we can actually write $45 = (4 \times 9) + (4 + 5)$. So to show that $3 \mid 45$ all we need to show is that $3 \mid (4 \times 9) + (4 + 5)$. Since $3 \mid 9$ we know $3 \mid (9 \times 4)$ by Property 3 of divisibility which we covered earlier, and by our hypothesis we know that $3 \mid (4 + 5)$. Therefore Property 1 of divisibility tells us that $3 \mid ((4 \times 9) + (4 + 5))$ and so $3 \mid 45$ as desired!!

Now let's generalize our findings from Example 3.3!

Example 3.3

Prove that if x is a two digit number such that the sum of the digits of x is a multiple of 3 then $3 \mid x$.

Solution:

Assume that x is a two digit number, further assume that the digits of x are a and b. That is, $x = (10 \times a) + (1 \times b)$ Recall from Example 3.3 that we can equivalently rewrite $(10 \times a) + b$ as $(9 \times a) + (a + b)$. The key piece of information we need to use is that the sum of the digits of x is a multiple of 3, equivalently this means $3 \mid (a + b)$.

Similar to what we did in Example 3.3, we know that $3 \times 3 = 9$ so $3 \mid 9$ and thus by Property 3 of divisibility we have that $3 \mid (9 \times a)$. Putting this together we have that $3 \mid (a + b)$ and $3 \mid (9 \times a)$, so Property 1 of divisibility tells us that $3 \mid (9 \times a) + (a + b)$. Therefore $3 \mid x$!

Exercise 3.3

Prove that if x is a THREE digit number whose digits add up to a multiple of 3, then $3 \mid x$.

Stop and Think

How could we generalize this proof in the case where x has 4 digits? 5 digits?... or has any number of digits?

Stop and Think

How would we prove the following; if $3 \mid x$ then the sum of the digits of x is a multiple of 3.

Rule for Division by 4

Let x be a whole number. If the last two digits (tens and ones) of x make up a number which is divisible by 4, then $4 \mid x$

For example $4 \mid 92912$ since $4 \mid 12$.

Exercise 3.4

Use the Rule for Division by 4 to determine if the following are true or false- check your answers with a calculator.

1. $4 \mid 342$
2. $4 \mid 1342$
3. $4 \mid 29739484$

Let's show that this rule holds for a specific 3 digit number;

Example 3.4

Show that if $4 \mid 84$ then $4 \mid 384$

Solution: Since $4 \times 21 = 84$ we know that $4 \mid 84$. Our goal is to show that $4 \mid 384$. We will begin by rewriting 384 as a sum of the products of its digits times its place value. Doing this we get $384 = (100 \times 3) + (10 \times 8) + (1 \times 4)$.

Since $4 \mid 84$, we know that $4 \mid (10 \times 8) + (1 \times 4)$ so all we need to do is find a way to show that $4 \mid (100 \times 3)$

Luckily this is very easy if we first notice that $4 \mid 100$! Since $4 \mid 100$ we know $4 \mid (100 \times 3)$ by Property 3 of divisibility which we covered earlier.

Therefore by Property 1 of divisibility since $4 \mid (10 \times 8) + (1 \times 4)$ and $4 \mid (100 \times 3)$ then $4 \mid ((100 \times 3) + (10 \times 8) + (1 \times 4))$ which gives us that $4 \mid 384$ as desired!

Exercise 3.5

Generalize our findings from Example 3.4. That is, prove that if x is a four digit number where 4 divides the number in the last two digits of x then $4 \mid x$.

Rule for Division by 5

Let x be a whole number. if the last digit of x is a 0 or a 5 then $5 \mid x$

The proof for this rule is very similar to our proof of the Rule for Division by 2 since we are only concerned about the ones digit of our whole number x!

Stop and Think

How would you prove that if the last digit of a whole number x is a 0 or a 5 then $5 \mid x$

Rule for Division by 6

Let x be a whole number. If x is an even number whose digits add up to a multiple of 3 then $6 \mid x$.

6 is the first number in our list which we can express as the product of two numbers which have no common factors other than ± 1, and because of this we can notice that the rule for divisibility by 6 is actually just the combination of the Rule for Division by 2 and the Rule for Division by 3!

Stop and Think

How could you come up with a Rule for Division by 12? 18? 24?

Rule for Division by 7

Let x be a whole number. If the difference between two times the ones digit of x and the remaining part of x is divisible by 7 then $7 \mid x$.

Example 3.6

Check if $7 \mid 798$

Solution: The ones digit of 798 is 8, and so 2 $\times$ the ones digit of 798 is $2 \times 8 = 16$.
The remaining part of 798 is 79. Now, take the difference between 79 and 16, doing this we get $79 - 16 = 63$. Here, the difference value obtained is 63, which is a multiple of 7 (i.e., 9 x 7 = 63). Thus, the given number 798 is divisible by 7.

Exercise 3.7

Use the Rule for Division by 7 to determine if the following are true or false- check your answers with a calculator.

1. $7 \mid 21$
2. $7 \mid 5469$
3. $7 \mid 346$

Rule for Division by 8

Let x be a whole number. If the the last three digits (hundreds, tens and ones) of x make up a number which is divisible by 8, then $8 \mid x$.

This rule is very similar to our rule for Division by 4 except we are now considering the 3 last place values rather than just the last 2. For example $8 \mid 92800$ since $8 \mid 800$.

Exercise 3.8

Use the Rule for Division by 8 to determine if the following are true or false- check your answers with a calculator.

1. $8 \mid 3200$
2. $8 \mid 12983810$

Rule for Division by 9

Let x be a whole number. If the digits of x add up to a multiple of 9, then $9 \mid x$.

Exercise 3.9

Use the Rule for Division by 9 to determine if the following are true or false- check your answers with a calculator.

1. $9 \mid 348293$
2. $9 \mid 12983810$

Divisibility Rules:

For all the following statements let x be a whole number.

1. **Rule for Division by 1:** 1 divides all numbers.
2. **Rule for Division by 2:** If the last digit of x is a 0, 2, 4, 6, or an 8, then $2 \mid x$.
3. **Rule for Division by 3:** If the digits of x add up to a multiple of 3, then $3 \mid x$.
4. **Rule for Division by 4:** If the last two digits (tens and ones) of x make up a number which is divisible by 4, then $4 \mid x$.
5. **Rule for Division by 5:** If the last digit of x is a 0 or a 5, then $5 \mid x$.
6. **Rule for Division by 6:** If x is an even number whose digits add up to a multiple of 3, then $6 \mid x$.
7. **Rule for Division by 7:** If the difference between two times the ones digit of x and the remaining part of x is divisible by 7, then $7 \mid x$.
8. **Rule for Division by 8:** If the the last three digits (hundreds, tens and ones) of x make up a number which is divisible by 8, then $8 \mid x$.
9. **Rule for Division by 9:** If the digits of x add up to a multiple of 9, then $9 \mid x$.

***Note** the converses of all these statements are true as well!